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COVERING, DOMINANCE, AND INSTITUTION FREE PROPERTIES OF SOCIAL CHOICE*

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ABSTRACT

This paper shows that different institutional structures for aggregation of preferences under majority rule may generate social choices that are quite similar, so that the actual social choice may be rather insensitive to the choice of institutional rules.

Specifically, in a multidimensional setting, where all voters have strictly quasi concave preferences, it is shown that the "uncovered set" contains the outcomes that would arise from equilibrium behavior under three different institutional settings. The three institutional settings are two candidate competition in a large electorate, cooperative behavior in small committees, and sophisticated voting behavior in a legislative environment where the agenda is determined endogenously.

Because of its apparent institution free properties, the uncovered set may provide a useful generalization of the core when a core does not exist. A general existence theorem for the uncovered set is proven, and for the Downsian case, bounds for the uncovered set are computed. These bounds show that the uncovered set is centered around a generalized median set whose size is a measure of the degree of symmetry of the voter ideal points.

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1. Introduction

Recent results in social choice theory for multidimensional choice spaces have shown not only the genericity of non-existence of core alternatives (Plott [1967], Rubinstein [1979], Schofield [1983]), but also, the genericity of global cycle sets (McKelvey [1976], [1979], Schofield [1978], [1983]). The conclusions drawn from these results have led to renewed interest in the role of rules and institutions in determining social outcomes (see e.g., Shepsle and Weingast [1981], and Ordeshook and Shepsle [1982]). The result has been a growing body of literature which explicitly models the institutional structure, and then looks for game theoretic equilibria conditional on the strategies that are implied by that structure (e.g., Shepsle [1979], Kramer [1972], [1977], McKelvey, Ordeshook and Winer [1978], and Ferejohn et al [1980], [1984]). While these directions are a healthy development in the field, one is left with the impression that, in the absence of core alternatives, little can be said about social choice that is institution independent. We argue here that such a conclusion is not warranted.

While it may be necessary to model explicitly the institutional structure if one wants to obtain exact solutions, there appears to be enough commonality among the outcomes selected under different institutional arrangements so that nontrivial bounds on social choice can be determined which hold under several different institutions. Miller [1980], in a very insightful paper, argued that for finite alternative spaces, the "uncovered set" performs just such a function. The covering relation, as defined by Miller, provides a method for transitively ordering alternatives for social choice. Miller argues that the resulting set of "uncovered" alternatives serves as a general solution set for majority voting games. Specifically, he shows that under a variety of institutional settings, game theoretic behavior by participants leads to outcomes in the uncovered set.

The covering relation is closely related to the notion of dominance between strategies in two person games. As such, it has also been studied by other authors in slightly different versions. McKelvey and Ordeshook [1976] define an "admissibility relation" which is quite similar to the covering relation, and show the connection between the admissible set and outcomes resulting from two candidate competition. Fishburn [1977] defines and investigates the normative properties of a set based on a dominance like relation. Richelson [1980] shows the connection between Fishburn's set and the uncovered set, and introduces a choice set based on a more natural definition of dominance. More recently, Shepsle and Weingast [1984] have applied and extended Miller's results to show how the covering relation can be used to get bounds on agenda reachable outcomes in multidimensional

choice spaces.

This paper studies the dominance and covering relations in a setting that is of more immediate interest to economists and political theorists. Namely we assume multidimensional choice spaces with quasi concave preferences. This is more general than Miller's set up in terms of the assumptions on the alternative set and the social order, because we allow for infinite alternative sets, social indifference between alternatives and more general social choice functions than majority rule. But it is less general than Miller in the assumptions on preferences, because restrictions are placed on the allowable preference profiles. These are the standard continuity and convexity assumptions used by economists, however.

The results here are similar in spirit to those of Miller, and reaffirm the importance of the uncovered and undominated sets. Specifically, it is shown that the uncovered and undominated sets <u>exist</u> in this more general setting, that the undominated set is contained in the uncovered set, and that the uncovered set contains, as subsets, the solutions which arise as game theoretic equilibria in several different institutional settings. Namely, the uncovered set contains as a subset <u>any</u> Von Neumann Morgenstern Solution. It contains the support set of any mixed strategy equilibrium to the related two candidate competition game. Finally, it contains the outcomes that result from sophisticated voting when agendas are determined endogenously.

It thus appears that the uncovered set may be an important generalization of the core when core points do not exist. Namely, it allows us to give restrictions on what outcomes might arise as the result of game theoretic or "incentive compatible" behavior of individuals, under a number of different institutional mechanisms. It is thus of interest to determine the properties and relative size of the uncovered set.

A final section of the paper obtains bounds on the uncovered set for the case of "Downsian," or "Euclidian based preferences." These results confirm a conjecture of Miller's. Namely, the uncovered set is a centrally located set which collapses to the core when one exists, and which is smaller the closer the configuration of preferences is to having a core.

Section 2 introduces assumptions on voter preferences and derives some basic properties of the social preference relation. Section 3 defines the covering and dominance relations, giving properties of both. Section 4 gives the basic existence theorems for the uncovered and undominated sets. Sections 5-7 show the relation between the three different institutional mechanisms and the uncovered set. Section 8 derives bounds on the uncovered set under the assumption of "Downsian" preferences, and Section 9 gives concluding comments.

2. Assumptions: Voters and the Social Preference Order

We assume there is a finite set, N, of voters, a convex set $X \subseteq \mathbb{R}^m$ of alternatives, and for each $i \in N$, a weak order, $R_i \subseteq X \times X$

representing i's preferences. So each R_i is reflexive, complete, and transitive. We let P_i and I_i denote the asymmetric and symmetric parts of R_i, respectively.

We assume X is endowed with the standard metric topology, and for any set $A \subseteq X$, we use the notation $\beta(A)$, $\bar{A}$, A^c, and A^o for the boundary, closure, complement, and interior of A, respectively. For any binary relation $Q \subseteq X \times X$, we use the notation $xQy \iff (x,y) \in Q$, and $Q(x) = \{y \in X \mid yQx\}$. Thus, Q can be viewed as a correspondence $Q: X \twoheadrightarrow X$. Also, we write $Q^1 = Q$, and for any integer $k > 1$, $Q^k = Q^{k-1} \circ Q$. We write Q^{-k} for the relation satisfying $xQ^{-k}y \iff yQ^kx$. So xQ^ky iff there is a k step path, via Q, from y to x, and $xQ^{-k}y$ iff there is a k step path, via Q^{-1} from y to x.

We now introduce the assumptions we make on individual preferences. Throughout, we always make the following assumption:

A0: (Continuous Preferences) For all $i \in N$, and all $x \in X$, $R_i(x)$ and $R_i^{-1}(x)$ are closed.

This assumption guarantees that each voter's preferences can be represented by a continuous utility function. In addition, to A0, we will make the following assumptions when they are needed.

A1: (Strict Quasi Concave Preferences) For all $i \in N$, and $x,y \in X$ with $y \in R_i(x)$, if $z = ty + (1-t)x$, with $0 < t < 1$, then $z \in P_i(x)$.

A2: (Compact Preferences) For all $i \in N$ and $x,y \in X$, $R_i(x)$ is compact.

Assumption A1 requires that for all voters, the set of points that are preferred or indifferent to any alternative, x, is always a convex set. Further, there can be no "thick" indifference curves. Assumption A2 requires that the set of points at least as good as x must be a compact set. Typical economic preferences on $\mathbb{R}^m_+$, where more is better, generally satisfy Assumption A1, but not A2, whereas the usual preferences assumed in political science, where there is an implicit budget constraint, and individuals can be satiated, would satisfy both A1 and A2. In particular the quadratic based preferences of Davis and Hinich [1968] satisfy both A1 and A2.

We now define the social preference order based on a set $\underline{W} \subseteq 2^N$ of coalitions in N. We call $\underline{W}$ the set of <u>winning coalitions</u>. Throughout, we always assume

B0: ($\underline{W}$ is monotonic) $C \in \underline{W}$ and $C \subseteq C' \Rightarrow C' \in \underline{W}$ and ($\underline{W}$ is proper) $C \in \underline{W} \Rightarrow C^c \notin \underline{W}$.

So every superset of a winning coalition is winning, and a coalition and its complement cannot both be winning. Another condition, which will be used when needed is

B1: ($\underline{W}$ is strong) $C \notin \underline{W} \Rightarrow C^c \in \underline{W}$.

We define $\underline{B} = \{C \subseteq N | C^c \notin \underline{W}\}$ to be the set of <u>blocking</u> coalitions (clearly $\underline{W} \subseteq \underline{B}$), and $\underline{L} = 2^N - \underline{B}$ to be the <u>losing</u> coalitions. Note if

$\underline{W}$ satisfies B1, then $\underline{B} = \underline{W}$, and $\underline{L}$ and $\underline{W}$ partition 2^N. For any coalition $C \subseteq N$, we use the notation

$$xP_Cy \Leftrightarrow xP_iy \text{ for } \forall\, i \in C$$
$$xR_Cy \Leftrightarrow xR_iy \text{ for } \forall\, i \in C \qquad (2.1)$$

for the coalition preference relations, and

$$xPy \Leftrightarrow xP_Cy \text{ for some } C \in \underline{W}$$
$$xRy \Leftrightarrow xR_Cy \text{ for some } C \in \underline{B} \qquad (2.2)$$

for the social preference relation. Note that P and R are related in that P is the asymmetric part of R, but P_C is not necessarily the asymmetric part of R_C. Also, for $C \subseteq N$, and $x \in X$, $R_C(x)$ is the closure of $P_C(x)$ if the Assumption A1 on preferences is met (see Lemma 1 of Appendix).

Condition B0 requires only that the set of winning coalitions be monotonic and proper, and it allows for the existence of non-trivial (i.e., non-winning) blocking coalitions. Committee systems, bicameral systems, weighted voting, and α-majority rule are all examples of systems whose winning coalitions satisfy B0. Condition B1 requires that there cannot be non-trivial blocking coalitions. So every coalition must be either winning or losing. Systems satisfying B1 must always have rules for "breaking ties." Majority rule with n odd, or majority rule with n even and a tiebreaking chairman are examples (but not the only examples) of systems satisfying B1.

Throughout this paper, we will <u>always</u> assume both conditions A0 and B0. Hence, they will not be explicitly stated in the assumptions of any of the theorems. The other assumptions will sometimes be required, and they will be identified when they are needed.

In the appendix, we prove a number of properties of the social order, P. The most important of these are listed here.

1. (Lemma 4): Under A1, for all $k = \pm 1, \pm 2, \ldots,$ both P^k and R^k are lower hemi continuous.

2. (Lemma 5,6): Under A1, for all $x \in X$,

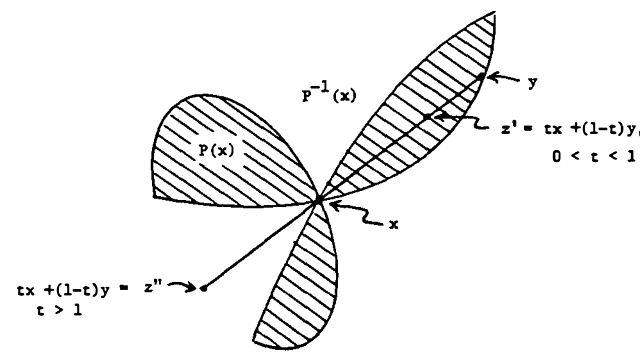
   (a). $P(x)$ is starlike about x, and for all $y \in P(x)$ and $z = ty + (1 - t)x$, with $t < 0$, $z \in P^{-1}(x)$.

   (b). Consequently, if $P(x) \neq \phi$, for every neighborhood, $N(x)$ of x, $N(x) \cap P(x) \neq \phi$ and $N(x) \cap P^{-1}(x) \neq 0$.

3. (Lemma 7): Under A1 and B1, for all $x \in X$, if $P(x) \neq \phi$, then $I(x) = \beta(P(x)) = \beta(P^{-1}(x))$. So $R(x) = \overline{P(x)}$.

The first property establishes continuity properties of the social order. Lower hemi continuity insures that points in $P^k(x)$ (or $R^k(x)$) cannot suddenly disappear with arbitrarily small changes in x. The second property is illustrated in Figure 1. The first part says that if $y \in P(x)$ and $z = ty + (1 - t)x$, then if z is on the line

segment connecting x and y (i.e. $0 < t < 1$), then $z \in P(x)$, whereas if z is on the ray emanating from x, in the opposite direction from y, (i.e., $t < 0$), then $z \in P^{-1}(x)$. From this result, part (b) follows immediately. Namely, whenever $P(x)$ is empty, then any neighborhood of x must contain points both of $P(x)$ and of $P^{-1}(x)$. The last property shows that when the social order is generated by a strong game, then $R(x)$ is the closure of $P(x)$.



<u>Figure 1</u>
Illustration of P(x) for three voters
Note P(x) is starlike about x
For any y, $z' \in P(x)$, and $z" \notin P(x)$

3. <u>The Dominance and Covering Relations</u>

We define two binary relations, which we call the dominance and covering relations, respectively. They are both variations of the <u>weak dominance</u> relation, $\overline{D} \subseteq X \times X$ which is defined as follows. For any $x, y \in X$,

$$x\bar{D}y \iff P(x) \subseteq P(y) \text{ and } R(x) \subseteq R(y). \tag{3.1}$$

The <u>dominance relation</u>, $D \subseteq X \times X$ is just the asymmetric part of $\bar{D}$. So, for any $x, y \in X$,

$$\begin{aligned} xDy &\iff x\bar{D}y \text{ \& } \sim y\bar{D}x \\ &\iff P(x) \subseteq P(y) \text{ and } R(x) \subseteq R(y), \\ &\qquad \text{with one inclusion strict.} \end{aligned} \tag{3.2}$$

The second relation we define is a slight variation of the dominance relation, and we call it the covering relation. It should be noted, however, that the definition we give is somewhat different than that of Miller [1982]. The <u>covering relation</u>, denoted $C \subseteq X \times X$ is defined by, for all $x, y \in X$,

$$xCy \iff x\bar{D}y \text{ \& } xPy. \tag{3.3}$$

The following propositions give properties of the dominance and covering relation and are proven in the appendix.

<u>Proposition 3.1</u> Both D and C are symmetric, irreflexive, transitive and acyclic. Further $C \subseteq D$.

<u>Proposition 3.2</u> Under Assumption A1, $\bar{D}(x)$ is closed for all x.

Under Assumptions A1 <u>and</u> B1, the definitions of dominance and covering simplify considerably, so that we need only verify one inclusion:

<u>Proposition 3.3</u> Under Assumptions A1 and B1, we have, for all $x, y \in X$

$$x\bar{D}y \iff P(x) \subseteq P(y) \iff R(x) \subseteq R(y)$$

$$xDy \iff P(x) \subseteq P(y) \iff R(x) \subseteq R(y)$$

$$xCy \iff \{x\} \cup P(x) \subseteq P(y) \iff \{x\} \cup P(x) \subset P(y)$$

The next result is a version of Miller's "two step" principle, which, under A1 and B1, also holds in the spatial framework developed here. Namely, x weakly dominates y if and only if it is not possible to reach y, in two steps, from x.

<u>Proposition 3.4</u> For any $y, x \in X$, $x \in P^{-2}(y) \Rightarrow \neg x\bar{D}y$. Under Assumptions A1 and B1, $x \in P^{-2}(y) \iff \neg x\bar{D}y$. Equivalently, $y \notin P^2(x) \iff x\bar{D}y$.

4. <u>The Undominated and Uncovered Sets</u>

The undominated and uncovered sets are defined to be the maximal elements of the dominance and covering relations, respectively. More formally, for any binary relation, $Q \subseteq X \times X$ and set $A \subseteq X$, the <u>maximal elements</u> of the relation Q in A are defined:

$$\underline{M}(Q, A) = \{x \in A \mid \forall\, y \in A,\ xQy \text{ or } \sim(yQx)\}$$

If Q is asymmetric (as are the covering and dominance relations), then we can write

$$\underline{M}(Q,A) = \{x \in A \mid \forall\, y \in A,\ \sim yQx\}.$$

Then we define the undominated set of X in A, written $\underline{UD}(X|A)$, and the uncovered set of X in A, written $\underline{UC}(X|A)$ by

$$\underline{UD}(X|A) = \underline{M}(D,A)$$

and

$$\underline{UC}(X|A) = \underline{M}(C,A)$$

We write $\underline{UD}(X)$ for $\underline{UD}(X|X)$ and $\underline{UC}(X)$ for $\underline{UC}(X|X)$, and refer to $\underline{UD}(X)$ and $\underline{UC}(X)$ as the undominated and uncovered sets of X, respectively. Since $C \subseteq D$, it follows easily that $\underline{UD}(X) \subseteq \underline{UC}(X)$.

Our first theorem establishes existence of the uncovered set and the undominated set. With quasi concave preferences (Assumption A1), to guarantee existence, it is sufficient to assume either that the alternative set is compact or that individual preferences are compact (i.e. individuals always have compact "preferred to" sets).

Theorem 1 Under Assumption A1, for any compact $A \subseteq X$, $\underline{UD}(X|A) \neq \emptyset$ and $\underline{UC}(X|A) \neq \emptyset$. If Assumption A2 is also satisfied, then for any $X \subseteq \mathbb{R}^m$, $\underline{UD}(X) \neq \emptyset$ and $\underline{UC}(X) \neq \emptyset$.

Proof: For any $A \subseteq X$, it follows easily that $\underline{M}(D,A) \subseteq \underline{M}(C,A)$. This follows directly by Prop 3.1, since $C \subseteq D$. But now, it suffices to prove the theorem for $\underline{M}(D,A)$ and $\underline{M}(D,X)$, since the corresponding results for $\underline{M}(C,A)$ and $\underline{M}(C,X)$ follow from the above inclusion.

Let $A \subseteq X$ be compact, and let $E \subseteq A$ be a chain under the partial ordering induced by $\bar{D}$. We will show that any such chain has an upper bound in A. Set $F = \bigcap_{x \in E} [\bar{D}(x) \cap A]$. If $F \neq \emptyset$, then pick $x^* \in F$. So $x^* \bar{D} x$ for all $x \in E$, hence x^* is an upper bound for E. If $F = \emptyset$, then $\{\bar{D}(x)^c | x \in E\}$ is an open cover of A, since by Prop 3.2, each $\bar{D}(x)$ is closed. But then there is a finite subset $G \subseteq E$ such that $\{\bar{D}(x)^c | x \in G\}$ is an open cover of A. By transitivity of $\bar{D}$, and the fact G is a chain, there is an $x^* \in G$ with $x^* \bar{D} x$ for all $x \in G$. Also by transitivity of $\bar{D}$, $\bar{D}(x^*) \subseteq \bar{D}(x)$ for all $x \in G$. But then $A \subseteq \bar{D}(x^*)^c$, so $\bar{D}(x^*) \cap A = \emptyset$, and a fortiori, $\bar{D}(x^*) \cap E = \emptyset$. I.e., we have $x^* \in E$ with $x \bar{D} x^*$ for no $x \in E$. But since E is a chain, we must have $x^* \bar{D} x$ for all $x \in E$. But then x^* is an upper bound for E. Thus, we have shown every chain $E \subseteq A$ has an upper bound in A hence, by Zorn's lemma, there is a maximal element in A, i.e., $\underline{M}(\bar{D},A) \neq \emptyset$. But $\underline{M}(D,A) = \underline{M}(\bar{D},A)$, since D is the asymmetric part of $\bar{D}$. Hence $\underline{M}(D,A) \neq \emptyset$.

Now if A2 is satisfied, and X is any (not necessarily compact) subset of $\mathbb{R}^m$ then let $E \subseteq X$ be a chain. Pick arbitrary $x_0 \in E$, and set $E^+ = \{y \in E | yDx_0\}$, $E^- = \{y \in E | x_0Dy\}$. Now E^+ is a chain contained in $\bar{D}(x_0)$, which, by A2, is compact. By the same proof as above, E^+ has an upper bound, say x^*, but x^* is also an upper bound for E, by transitivity. Hence every chain $E \subseteq X$ has an upper bound. Applying Zorn's lemma again, $\underline{M}(\bar{D},X) = \underline{M}(D,X) \neq \emptyset$.

Q.E.D.

The two step principle can be extended to get upper and lower bounds on the uncovered set in terms of the set of points that are reachable in two steps from all other alternatives. When both A1 and B1 are met, then with the exception of points of closure, the uncovered set (and undominated set) is characterized as the set of points which are reachable in two steps, via P, from every other point in X. In this case, it follows that $\overline{\underline{UD}(X)} = \overline{\underline{UC}(X)}$, so the two sets are identical except for points of closure.

Proposition 4.1: In general

$$\bigcap_{y \in X} P^2(y) \subseteq \underline{UD}(X) \subseteq \underline{UC}(X) \subseteq \bigcap_{y \in X} R^2(y).$$

If A1 and B1 are satisfied, and $P(z) \neq \emptyset$ for all $z \in X$, then

$$\bigcap_{y \in X} P^2(y) \subseteq \underline{UD}(X) \subseteq \underline{UC}(X) \subseteq \overline{\bigcap_{y \in X} P^2(y)}.$$

When A1 and B1 are met, the above proposition gives a potential "brute force" method for computing $\overline{\underline{UC}(X)}$ up to any desired degree of accuracy. One could simply check whether $x \in P^2(y)$ for all y on some fine enough grid in X.

The next two results give properties of the uncovered set and undominated set which are useful later in dealing with endogenous agendas:

Proposition 4.2 Under Assumptions A1 and B1, for all $A \subseteq X$ with either $X = A$ or A compact, and for all $x \in A$, if $x \notin \underline{UD}(X|A)$ then $\exists\, y \in \underline{UD}(X|A)$ with yDx. Similar results hold for $\underline{UC}(X|A)$.

Proposition 4.3 For any collection $\{x^j\} \subseteq X$ with $\bigcap_j P(x^j) \neq \emptyset$, we have $\bigcap_j P(x^j) \cap \underline{UD}(X) \neq \emptyset$, and $\bigcap_j P(x^j) \cap \underline{UC}(X) \neq \emptyset$.

It should be noted that for all $x \in \bigcap_j P(x^j)$, either $x \in \underline{UD}(X)$ or there is a $y \in \bigcap_j P(x^j) \cap \underline{UD}(X)$ with yDx. (Similarly for $\underline{UC}(X)$).

5. Small Committees

Small committee behavior is usually modeled as a cooperative game, in characteristic function form. This provides a good model of the social outcomes that would occur in small groups operating with unstructured rules of procedure. In this section, we define a simple characteristic function form game, without sidepayments, whose winning coalitions are $\underline{W}$. We then show that every Von-Neumann-Morgenstern solution to this game is included in the uncovered set.

Before proceeding, we warn the reader of some terminological ambiguity which leads to considerable confusion. The word "dominance" is used in game thoery with two distinctly different meanings. In one usage, it refers to the relation between two strategies in a non-cooperative game. Here, one strategy dominates another if it is better no matter what strategy the opponents adopt. We will see, in the next section, that this usage of the word coincides with our relation D, which we have thus called the dominance relation. The second usage of "dominance" is in cooperative games, where it refers to a relation between two strategies adopted by, perhaps, different coalitions. Here, one strategy (or alternative) "dominates" another if there is some coalition which both prefers and can unilaterally insure the first over the second. This second usage of the word "dominance" is the usage which is meant in this section of the paper. Since we will have occasion to use "dominance" in both of its meanings in this paper, when there is the possibility of confusion, we will use the terms "Φ-dominance" and "v-dominance" to refer to the non-cooperative and cooperative usages of the term. Here Φ refers to the payoff function of the corresponding non-cooperative game, and v to the characteristic function of the corresponding cooperative game.

We define the characteristic function, $v:2^N \to 2^X$ for a simple game without sidepayments by:

$$v(C) = X \text{ if } C \in \underline{W}$$

$$v(C) = \emptyset \text{ if } C \notin \underline{W}$$

For any $x,y \in X$ and $C \subseteq N$, we say x <u>v-dominates</u> y <u>via</u> C, written $x \underset{C}{\succ} y$, iff xP_Cy and $x \in v(C)$. We say x <u>v-dominates</u> y, written $x \succ y$ iff $x \underset{C}{\succ} y$ for some $C \subseteq N$. It is trivial to verify that for the simple characteristic function defined above, $x \underset{C}{\succ} y$ iff xP_Cy and $x \succ y$ iff xPy.

A <u>Von-Neumann-Morgenstern Solution</u> is defined to be any set $K \subseteq X$ satisfying

(a) $\forall\, x,y \in K$, $\sim(x \succ y)$ (internal stability)

(b) $\forall\, x \notin K$, $\exists y \in K$ s.t. $y \succ x$ (external stability).

<u>Theorem 2</u>: Let A1 and B1 be met, and let K be a VNM solution. Then $K \subseteq \underline{UC}(X)$.

<u>Proof</u>: We first show that for any $x \in K$, $z \notin K$, that $\sim zDx$. To prove this, by external stability it follows that $\exists y \in K$ with yPz. By openness of $P(z)$, it follows there is a neighborhood $N(y)$ of y such that $y'Pz$ for any $y' \in N(y)$. By internal stability, xIy. By Lemma 8, $P(x) \neq \emptyset$. By Lemma 7, we can find $y' \in N(y)$ with $y' \in P^{-1}(x)$. But then $y' \in P^{-1}(x)$ and $z \in P^{-1}(y')$, so $z \in P^{-2}(x)$. By Prop 3.4, $\sim zDx$.

To show that $K \subseteq \underline{M}(C,X)$, note that from the first paragraph, whenever $x \in K$, $z \notin K$, we have $\sim zDx$. But since $zCx \Rightarrow zDx$, it follows that $\sim zCx$. But also, if $x \in K$, and $z \in K$, then by internal stability, $\sim zPx$ so $\sim zCx$. Thus for all $x \in K$, $z \in X$, $\sim zCx$. Hence $x \in \underline{M}(C,X)$, so $K \subseteq \underline{M}(C,X)$.

Q.E.D.

Typically, a Von Neumann Morgenstern Solution is not unique. In addition to finite solutions (where it is possible to view each alternative in K as the proposal of some winning coalition), there may be an infinite number of "discriminatory" solutions. The discriminatory solutions each contain an infinite number of alternatives, and in some games (for example majority rule "divide the dollar" games) their union covers the whole set of Pareto Optimals. It is worth emphasizing that the above theorem holds for <u>any</u> Von Neumann Morgenstern Solution. Hence it follows from the above result that the union of all Von Neumann Morgenstern solutions (i.e., the set of all points which could result from some VNM solution) is also in the uncovered set.

6. Two Candidate Competition

The covering relation also turns out to have a close connection with models of two candidate competition. We can model two candidate competition as a two person, zero sum game, with the candidates as players, who compete for the votes of the electorate through the policy positions they adopt. Thus, candidate strategies consist of an announcement of a policy position that they will adopt if elected, and then voters are assumed to vote for the candidate whose policy position they prefer. Formally, we define a two player, symmetric zero sum game as follows: the strategy spaces for both players will be $S_1 = S_2 = X$. The payoff function, Φ: $S_1 \times S_2 \rightarrow R$ for player 1 is defined by, for $\underline{s} = (s_1, s_2) \in S_1 \times S_2$

$$\Phi(\underline{s}) = \begin{cases} 1 \text{ if } s_1 P s_2 \\ -1 \text{ if } s_2 P s_1 \\ 0 \text{ otherwise} \end{cases} \qquad (6.1)$$

Since the game is zero sum, the payoff to player 2 is just $-\Phi(\underline{s})$.

It is easily shown that the above game has a pure strategy equilibrium if and only if there is a majority rule core point (i.e., iff $\underline{M}(P,X) \neq \emptyset$). Hence, it follows from Plott's [1967] Theorem that there will generally not exist pure strategy equilibria to the above

game. Despite the fact that there are no pure strategy equilibria, one can still make some statements about the policy outcomes which might occur. There are two ways of doing this: one is to look for mixed strategy equilibria, and the other is to use Farquharson's [1969] concept of "sophisticated" strategies.

Before proceeding, we recall the definition of domination of two strategies for a non-cooperative game, and show that it corresponds to the dominance relation, D, defined earlier. Given two strategies, s_1, $t_1 \in S_1$ for player 1, we say that s_1 <u>Φ-dominates</u> t_1 if, for all choices of strategy $s_2 \in S_2$ by player 2,

$$\Phi(s_1, s_2) \geq \Phi(t_1, s_2) \qquad (6.2)$$

with strict equality for some $s_2 \in S_2$. But, from (3.2),

$$s_1 D t_1 \iff \begin{array}{l} \text{(a)} \;\; P(s_1) \subseteq P(t_1) \text{ and } R(s_1) \subseteq R(t_1) \;\; \text{and} \\ \text{(b)} \;\; P(s_1) \subset P(t_1) \text{ or } R(s_1) \subset R(t_1) \end{array} \qquad (6.3)$$

However, (a) holds iff

$$\begin{aligned} & (\forall s_2 \in X)\ (s_2 P s_1 \Rightarrow s_2 P t_1) \text{ and } (s_2 R s_1 \Rightarrow s_2 R t_1) \\ \iff & (\forall s_2 \in X)(\Phi(s_1, s_2) = -1 \Rightarrow \Phi(t_1, s_2) = -1) \text{ and} \\ & (\Phi(s_1, s_2) \leq 0 \Rightarrow \Phi(t_1, s_2) \leq 0) \\ \iff & (\forall s_2 \in X)(\Phi(s_1, s_2) \geq \Phi(t_1, s_2)) \end{aligned} \qquad (6.4)$$

Similarly (b) holds iff

$$(\exists s_2 \in X)(\Phi(s_1,s_2) > \Phi(t_1,s_2)). \qquad (6.5)$$

Together, (6.4) and (6.5) yield (6.2). Thus, $s_1 D t_1$ if and only if s_1 Φ-dominates t_1 in the two person non-cooperative game of (6.1).

Farquharson's analysis of sophisticated behavior is based on the idea that players will eliminate from consideration any strategies that are game theoretically dominated (i.e., Φ-dominated). The remaining strategies are called admissible (or primarily admissible). Once the dominated strategies have been eliminated, a reduced game results, and strategies which were previously not dominated may now be dominated. Players can now eliminate strategies from the reduced game. The remaining strategies are called secondarily admissible, and the resulting game is called the second reduction. Proceeding in this fashion, one arrives eventually at ultimately admissible, or sophisticated strategies, which are those which survive all successive reductions. It is reasonable to think that when there is no pure strategy equilibrium, that candidates should confine themselves to ultimately admissible strategies (or at least to admissible strategies). The following theorem shows that any sophisticated strategy as well as the support of any mixed strategy equilibrium must be inside the uncovered set.

Theorem 3:

(a) All admissible strategies (and hence all ultimately admissible strategies) for the game (6.1) are included in $\underline{UC}(X)$.

(b) If $\lambda : X \to \underline{R}^m$ is a mixed strategy equilibrium for the game (6.1), then the support for λ, supp(λ) must satisfy supp(λ) $\subseteq \underline{UC}(X)$.

Proof: Part (a) follows directly from the fact that the relation D corresponds to Φ-dominance in the game (6.1). Thus the primarily admissible strategies are included in $\underline{UD}(X)$, which are in turn included in $\underline{UC}(X)$.

Part (b) follows from the well known fact that a mixed strategy equilibrium can not put positive measure on the set of dominated strategies.

Q.E.D.

On the question of existence of mixed strategy equilibria see Kramer [1978] and Rosker [ ], who show that for games similar to that of (6.1), mixed strategy equilibria will exist.

In the context of "Downsian," or "Euclidian based preferences," McKelvey and Ordeshook [1976] studied properties of a set very similar to the uncovered set (called the admissible set in their paper), and showed that it contains the mixed strategy solutions and is a subset of the convex hull of the "partial medians." Unfortunately, this set is frequently not very restrictive. Under similar assumptions on preferences, the uncovered set, on the other hand can frequently be quite restrictive, as is shown in the final section of this paper. Hence, Theorem 3 can be used to get nontrivial bounds on the support set of the mixed strategy solution to the candidate competition game.

7. Sophistication and Endogenous Agendas

The third institutional setting we consider is a legislature operating under a set of parliamentary rules, where the motions on the agenda are generated endogenously. We consider only amendment type procedures, where voters vote in a sophisticated fashion. Some preliminary definitions are in order.

We first define versions of the dominance and covering relations which apply to a subset of $A \subseteq X$. They are defined in the natural way. Thus, for any $x, y \in X$,

$$x\bar{D}_A y \iff [P(x) \cap A \subseteq P(y) \cap A] \text{ and } [R(x) \cap A \subseteq R(y) \cap A] \qquad (7.1)$$

and D_A is defined to be the asymmetric part of $\bar{D}_A$. I.e.,

$$xD_A y \iff x\bar{D}_A y \text{ and } \sim y\bar{D}_A x \qquad (7.2)$$

Also, we define

$$xC_A y \iff x\bar{D}_A y \text{ and } xPy \qquad (7.3)$$

For any $A, B \subseteq X$, the <u>undominated set of A in B</u>, written $\underline{UD}(A|B)$, is $\underline{M}(D_A, B)$, and the <u>uncovered set of A in B</u>, written $\underline{UC}(A|B)$, is $\underline{M}(C_A, B)$. Again we use the notation $\underline{UD}(A) = \underline{UD}(A|A)$ and $\underline{UC}(A) = \underline{UC}(A|A)$.

The following properties are easily verified. For any $A, B \subseteq X$, and $x, y \in X$

a) if $A \subseteq B$, then $x\bar{D}_B y \Rightarrow x\bar{D}_A y$ I.e., $A \subseteq B \Rightarrow \bar{D}_B \subseteq \bar{D}_A$

b) if $B \subseteq A$, and $x, y \in B$, $xC_A y \Rightarrow xD_A y$. I.e., $C_A \cap B \subseteq D_A \cap B$.

c) if $B \subseteq A$, $\underline{UD}(A|B) \subseteq \underline{UC}(A|B)$

Now an <u>agenda based on the amendment process</u> is simply a t-tuple $(x_1, \ldots, x_t)$ together with a voting tree, as illustrated in Figure 2. x_1 is called the <u>status quo</u>, x_2 the <u>motion</u>, x_3 the <u>amendment</u>, x_4 the <u>amendment to the amendment</u>, etc.

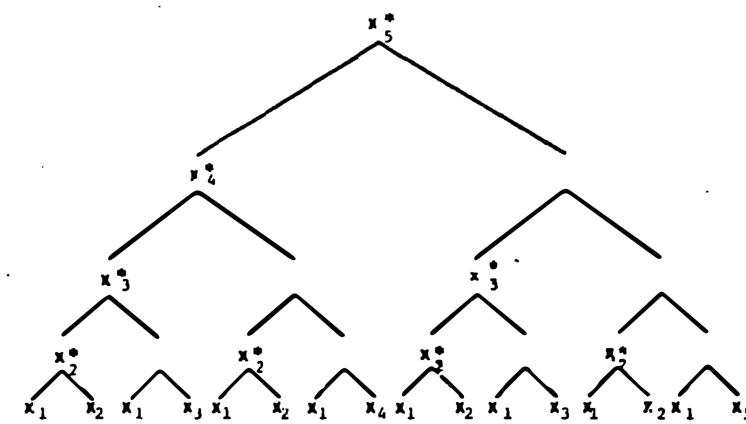


FIGURE 2 ILLUSTRATION OF THE AGENDA $(x_1, x_2, \ldots, x_n)$ WITH SOPHISTICATED EQUIVALENT $(x_1^*, x_2^*, \ldots, x_n^*)$

Given an agenda based on the amendment process, $(x_1, x_2, \ldots, x_t)$, we define its <u>sophisticated equivalent</u>, $(x_1^*, x_2^*, \ldots, x_t^*)$ as follows:

(a) $x_1^* = x_1$

(b) for $1 < i \leq n$ $\quad x_i^* = \begin{cases} x_i \text{ if } x_i \, P x_j^* \text{ for } \forall j < i \\ x_{i-1}^* \text{ otherwise.} \end{cases}$ (7.4)

The i^{th} element, x_i^*, of the sophisticated equivalent is called the sophisticated equivalent of node i and represents the outcome that would occur for the agenda $(x_1,\ldots,x_i)$ if no further amendments are introduced, and all voters vote sophisticatedly.

As discussed in the previous section, Farquharson's [1969] notion of sophisticated behavior amounts to successive elimination of dominated strategies. However, in this case, the underlying game is an n-person non-cooperative game, where individual strategy spaces are possible instructions as to how to vote at each node in the voting tree, and where the payoff function is the payoff that the voter gets from the final node which is reached, given an n-tuple of voter strategies. We will not write out the payoff function of this game formally, but results of McKelvey and Niemi [1978], and Gretlein [1983] show that sophisticated voting in this game is equivalent to the type of behavior described by (7.4). I.e. all voters vote at early nodes on the basis of what they realize will happen at successive nodes.

We now consider a fixed agenda $\underline{x} = (x_1,\ldots,x_t)$ and set $A = \{x_1,x_2,\ldots,x_k\}$. Then the following proposition extends a theorem of Miller [1980] to deal with a social ordering, P, which may or may not be antisymmetric (a "tournament" in Miller's terminology) over A:

Proposition 7.1: (Miller) In general, $x_t^* \in \underline{UC}(A)$. If P is antisymmetric on A, $x_t^* \in \underline{UD}(A)$.

While the above proposition shows that $x_t^* \in \underline{UC}(A)$, in general there may be an even more restrictive subset of $\underline{UC}(A)$, that contains x_t^*. See Banks [1985] for a characterization of the set of alternatives achievable by sophisticated agendas.

The above proposition relates to a fixed agenda, $(x_1,\ldots,x_t)$, where the alternatives, x_i, on the agenda are given exogenously. Further, the proposition only shows that the outcome resulting from sophisticated behavior is in the uncovered set of $A = \{x_1,x_2,\ldots,x_t\}$, where the covering relation, C_A, is defined with respect to the set A rather than with respect to X. It does not follow that x_t^* must be in the uncovered set of the whole space. Even if $A \cap \underline{UC}(X) \neq \emptyset$, we might have $\underline{UC}(A) \cap \underline{UC}(X) = \emptyset$. For example if $A = \{x_1,x_2\}$, where x_2Px_1, and $x_1 \in \underline{UC}(X)$ but $x_2 \notin \underline{UC}(X)$, then $\underline{UC}(A) = \{x_2\}$.

Even though Proposition 7.1 does not establish that x_t^* must be in the uncovered set of X, it does prove a powerful result which becomes useful in analyzing what might occur if there is any endogeneity to the process of agenda formation. Namely, the proposition shows that for any $x \in X$, no alternative $y \in X$ for which xCy can be the sophisticated outcome of an agenda which contains the two. If $A = \{x_1,x_2,\ldots,x_t\}$ is such an agenda, then $xCy \Rightarrow xC_Ay \Rightarrow y \notin \underline{UC}(A)$. So by the proposition, $y \neq x_t^*$. It follows that an individual can unilaterally quash any alternative in $C^{-1}(x)$ by introducing x on the agenda anywhere in the amendment process. Furthermore, the introduction of x on the agenda will have the desired effect (namely the elimination of $C^{-1}(x)$) regardless of whether the individual knows the other proposals on the agenda and regardless of whether other players are allowed to propose amendments subsequent to the introduction of x.

To formalize the above ideas, we first define $\underline{A}$ to be the set of all possible agendas. I.e.,

$$\underline{A} = \{\underline{x} \mid \underline{x} = (x_1, x_2, \ldots, x_t) \in X^t \text{ for some } t\}. \tag{7.5}$$

and for any $x \in \underline{A}$, we let $x^* \in X$ be the sophisticated outcome of the agenda x, i.e., if $\underline{x} = (x_1, \ldots, x_t)$, then $x^* = x_t^*$. Also, for any $\underline{x} = (x_1, \ldots, x_t) \in \underline{A}$, we write $A(\underline{x}) = \{x_1, \ldots, x_t\}$ for the set of proposals on the agenda $\underline{x}$, and for any $z \in X$, write $\underline{A}(z) = \{\underline{x} \in \underline{A} \mid z \in A(\underline{x})\}$ for the set of agendas containing z.

We can now define an n person game as follows: We let the set of players be $N = \{1, 2, \ldots, n\}$, for each $i \in N$, let $u_i\colon X \to \mathbb{R}$ be a continuous utility representation of R_i, and define the strategy space for player i to be $S_i = X$. Given choices $\underline{s} = (s_1, \ldots, s_n) \in \underline{S} = \prod_i S_i$, we define the payoff function $\Phi_i\colon \underline{S} \to \mathbb{R}$ by

$$\Phi_i(\underline{s}) = \inf_{\underline{x} \in \underline{A}(s_i)} u_i(x^*) \tag{7.6}$$

Thus, this game models the type of outcomes that might occur if alternatives on the agenda emerge endogenously and are then voted on in the following two step procedure: First, there is a period of agenda formation during which any individual has access to the floor and can introduce motions. During this stage, motions are made in an environment of incomplete information about how the amendments might

be ordered for voting or about what additional motions might arise. Once all motions have been made, they are placed on the agenda and voting occurs in a specific order, and according to the amendment process. During this stage, all voters are assumed to know the entire agenda and to vote sophisticatedly.

For any two strategy n-tuples, $\underline{s}$ and $\underline{t}$, we say that s and t are <u>payoff equivalent</u> if $\Phi_i(\underline{s}) = \Phi_i(\underline{t})$ for all i. We obtain the following result:

<u>Theorem 4</u>: If X is compact, and A1 and B1 are met, the game defined by (7.6) has a pure strategy equilibrium. Further, for any equilbrium, $\underline{s}$, there is a payoff equivalent strategy n-tuple, $\underline{t} = (t_1, \ldots, t_n) \in \underline{S}$ such that for any permutation $\varphi\colon N \to N$, and resulting agenda $(t_{\varphi(1)}, \ldots, t_{\varphi(n)})$, $t^*_{\varphi(n)} \in \underline{M}(C, X)$.

<u>Proof</u>: We first show that there is an equilibrium, $\underline{s}$. We let $W(s_i) = \{y \in X \mid y = x^* \text{ for some } \underline{x} \in \underline{A}(s_i)\}$. So

$$\Phi_i(\underline{s}) = \inf_{y \in W(s_i)} u_i(y) \tag{7.7}$$

We first note that, for any $x \in X$, $W(x) = X - C^{-1}(x)$. Miller's theorem gives us that $W(x) \subseteq X - C^{-1}(x)$ and to see $X - C^{-1}(x) \subseteq W(x)$, note for any $y \in X - C^{-1}(x)$, we have $\sim(xCy) \Rightarrow \sim(x\bar{D}y \text{ and } xPy)$ $\Rightarrow \sim(x\bar{D}y)$ or yRx. But if yRx, then the agenda (y,x) has sophisticated outcome y, an if $\sim(x\bar{D}y)$, then by Proposition 3.4, $x \in P^{-2}(y) \Rightarrow \exists z \in X$ with yPz and zPx. But then the agenda (z,y,x) has y as its sophisticated outcome. So we have proven

$W(x) = X - C^{-1}(x)$. But now, from Proposition 3.4, it follows easily that $X - C^{-1}(x) = P^2(x) \cup R(x)$. Hence, we have that

$$W(x) = P^2(x) \cup R(x) \tag{7.8}$$

Next, from Lemma 7, it follows that $\overline{P(z)} = R(z)$ whenever $P(z) \neq \Phi$, and that $\overline{P^2(x)} = R^2(x)$ whenever $P(x) \neq \Phi$. Hence $\overline{W(x)} = R^2(x)$ when $P(x) \neq \Phi$. By Lemma 4, $R(x)$, and hence $R^2(x)$ is a lower hemi continuous correspondence. But now,

$$\Phi_i(\underline{s}) = \inf_{y \varepsilon W(s_i)} u_i(y) = \inf_{y \varepsilon \overline{W(s_i)}} u_i(y) = \inf_{y \varepsilon R^2(s_i)} u_i(y) \tag{7.9}$$

But since $R^2(s_i)$ is lower hemi continuous and u_i is continuous, Φ_i is upper semi continuous on X. Hence, since X is compact, Φ_i attains a maximum. Hence an equilibrium exists to (7.9). It also follows that if $s = (s_1,\dots,s_n)$ is an equilibrium, then either $s_i \varepsilon \underline{M}(C,X)$ for all i, or there exists a "payoff equivalent equilibrium" in which all strategies are undominated. I.e., there are strategies $\underline{t} = (t_1,\dots,t_n)$ with $t_i \varepsilon \underline{M}(C,X)$ such that $\underline{t}$ is in equilibrium and $\Phi_i(\underline{t}) = \Phi_i(s)$ for all i. To see this, note that if for some $i \varepsilon N$, $s_i \notin \underline{M}(C,X)$, then by Proposition 4.2, $\exists t_i \varepsilon \underline{M}(C,X)$ with $t_i C s_i$. In this fashion, pick $\underline{t}$ with $t_i \varepsilon \underline{M}(C,X)$ for all $i \varepsilon N$. But by transitivity of C, $t_i C s_i \Rightarrow C^{-1}(s_i) \subseteq C^{-1}(t_i)$

$\Rightarrow X - C^{-1}(t_i) \subseteq X - C^{-1}(s_i) \Rightarrow W(t_i) \subseteq W(s_i) \Rightarrow \Phi_i(\underline{t}) \geq \Phi_i(\underline{s})$. But since $\underline{s}$ is an equilibrium, it must be that $\Phi_i(\underline{t}) = \Phi_i(\underline{s})$. So $\underline{t}$ is an equivalent equilibrium. We can proceed in this fashion to eventually obtain an equilibrium, $\underline{t}$, with all t_i in $\underline{M}(C,X)$. But now, the result that $t^*_{\phi(n)} \varepsilon \underline{M}(C,X)$ follows from the fact that all t_i are in $\underline{M}(C,X)$.

Q.E.D.

The game defined by (7.8) represents only <u>one</u> possible way of modeling endogenous formation of agendas, and the theorem then shows that all equilibria are payoff equivalent to equilibria in which all players restrict themselves to the uncovered set. Alternative methods of modeling the endogeneity of the proposals on the agenda would be indicated if legislators had more complete information during the motion making stage. We do not investigate those models here, but the same sort of considerations that drive motions into the uncovered set in the above model might also be expected to do so in other models. Proposition 4.3 shows that it is impossible to construct any agenda, $\underline{x} = (x_1,\dots,x_t)$ which will eliminate all uncovered points. I.e. for any such agenda, there always exists a proposal $x_{t+1} \varepsilon \underline{M}(C,X)$ which when added into the existing agenda will be the sophisticated outcome. And Miller's theorem together with Proposition 4.2 show the attractiveness of uncovered points in eliminating covered points. Based on these two forces we would conjecture that any model of endogenous agenda formation which allows all legislators the right to introduce motions will result in outcomes inside the uncovered set.

8. Size of the Uncovered Set

Given the results of the previous sections, it would be of interest to characterize the uncovered set as a function of the particular preference profile. Miller [1980] conjectures that the uncovered set will generally be a small, centrally located set. We have not been able to obtain results for general preference profiles satisfying A1 or A2. However, for the special case of Euclidian based preferences, we have been able to calculate bounds on the uncovered set. These results show that the uncovered set is indeed a "centrally located" set which collapses to the core when a core exists, and which is small if the preference configuration is "close" to having a core.

We assume in this section that $X = \mathbb{R}^m$, and that preferences are of the following form.

A3: (Euclidian based preferences) For all $i \in N$, $\exists x^i \in X$ such that for all $x,y \in X$

$$xRy \iff \|x - x^i\| \leq \|y - x^i\|.$$

These are the usual "Downsian" preferences, and x^i is called voter i's <u>ideal point</u>.

Additionally, we assume that the social preference order is generated by majority rule. Here we write $n = |N|$.

B2: (Majority rule) n is odd and $\underline{W} = \{C \subseteq N \mid |C| \geq \frac{n+1}{2}\}$

Clearly preferences satisfying A3 satisfy A0-A2, and if B2 is satisfied, then B0 and B1 are satisfied, hence all of the results of the previous sections apply here.

Now, for any $\alpha \in \mathbb{R}^m$ and $c \in \mathbb{R}$, write

$$H(\alpha,c) = \{x \in \mathbb{R}^m \mid x \cdot \alpha = c\}. \tag{8.1}$$

So $H(\alpha,c)$ is an $m - 1$ dimensional hyperplane in $\mathbb{R}^m$. The hyperplane $H(\alpha,c)$ is a <u>median hyperplane</u> if

$$|\{i \in N \mid x^i \cdot \alpha < c\}| \leq \frac{n}{2}$$
and
$$|\{i \in N \mid x^i \cdot \alpha > c\}| \geq \frac{n}{2}. \tag{8.2}$$

We let $\underline{H}$ denote the set of all median hyperplanes.

Next, for any $y \in X$ and $t \in \mathbb{R}^+$, let $\bar{B}(y,t)$ be the closed ball with center at y and radius t. So

$$\bar{B}(y,t) = \{x \in X \mid \|y - x\| \leq t\}. \tag{8.3}$$

We let $\underline{B}$ denote the set of all such balls which have non empty intersections with each $H \in \underline{H}$. I.e.

$$\underline{B} = \{\bar{B}(y,t) \mid y \in X,\ t \in \mathbb{R}^+, \text{ and for } \forall H \in \underline{H},\ H \cap \bar{B}(y,t) \neq \emptyset\}. \tag{8.4}$$

It is easily shown that under the assumptions made, there is a unique element, say $\bar{B}(\bar{y},\bar{t})$ of $\underline{B}$ satisfying, for $\forall \bar{B}(y,t) \in \underline{B}$, $\bar{t} \leq t$. The element $\bar{B}(\bar{y},\bar{t})$ of $\underline{B}$ is called the <u>Generalized Median Set</u>, or the "<u>yolk</u>" for short. It is the ball of minimum radius, which intersects every median hyperplane. The point $\bar{y}$ is referred to as the <u>generalized median point</u>. See Figure 3 for an illustration of this construction.

The yolk can be computed as a solution to a linear programming problem. Let $H_1, H_2, \ldots, H_k$ be the set of all median hyperplanes which contain at least two ideal points on the hyperplane. Assume α_i and c_i are the parameters describing H_i. Without loss of generality we can assume that $\| \alpha_i \| = 1$ for all i. So $H_i = \{x \in \mathbb{R}^m \mid x \cdot \alpha_i = c\}$. Since the distance from any $y \in \mathbb{R}^m$ to H_i is given by $| y \cdot \alpha_i - c_i |$, it follows that $\bar{y}$ and $\bar{t}$ are the solution to the following L.P.:

$$\begin{aligned} &\text{minimize } t \\ &\quad \text{s.t. } t \geq y \cdot \alpha_i - c_i \\ &\qquad\quad t \geq c_i - y \cdot \alpha_i \end{aligned} \qquad \text{for } \forall\, 1 \leq i \leq K$$

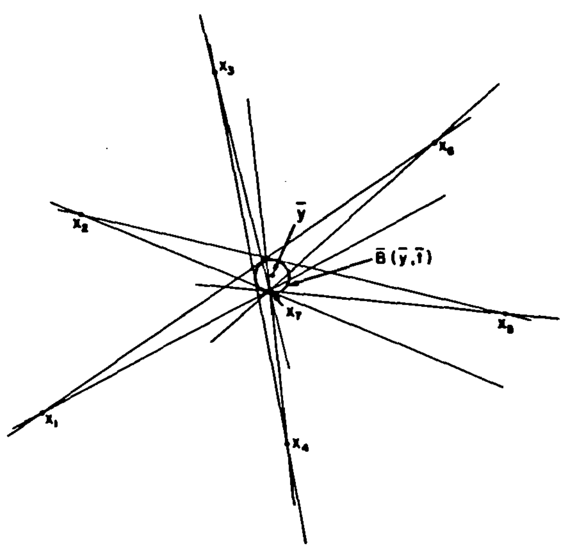


FIGURE 3 EXAMPLE OF THE CONSTRUCTION OF $\bar{B}(\bar{y},\bar{t})$ FOR A PREFERENCE CONFIGURATION OF SEVEN VOTERS IN TWO DIMENSIONS.

Next, given x^*, $y^* \in X$, and $t^* \in R$, define $C^m(x^*, y^*, t^*)$ to be the region bounded by the <u>m dimensional cardioid</u> which has <u>cusp</u> at x^*, <u>center</u> at y^*, <u>eccentricity</u> of t^*, and radius of $r^* = \|x^* - y^*\|$. See Figure 4. This is defined formally in Appendix B.

The following results are proven in the Appendices. See Figure 5 for an illustration.

<u>Proposition 8.1</u> Under Assumptions A3 and B2, for all $x \in X$

$$C^m(x, \bar{y}, -2\bar{t}) \subseteq P(x) \subseteq R(x) \subseteq C^m(x, \bar{y}, 2\bar{t})$$

Further, the following relation follows directly from the definition of the sets $C^m(x,y,t)$

<u>Proposition 8.2</u> Under Assumptions A3 and B2, for all $x \in X$, if $t = \|x - \bar{y}\|$, then

$$C^m(x, \bar{y}, 2\bar{t}) \subseteq \bar{B}(\bar{y}, t + 2\bar{t})$$
$$\bar{B}(\bar{y}, t - 2\bar{t}) \subseteq C^m(x, \bar{y}, -2\bar{t})$$

Putting together the above results, we get the following proposition:

<u>Proposition 8.3</u> Under Assumptions A3 and B2, then for all $x \in X$, setting $t = \|x - \bar{y}\|$, we have

$$\bar{B}(\bar{y}, t - 4\bar{t}) \subseteq D(x) \text{ and}$$
$$X - D^{-1}(x) \subseteq \bar{B}(\bar{y}, t + 4\bar{t}).$$

The proposition gives bounds on the set of points that cover and are

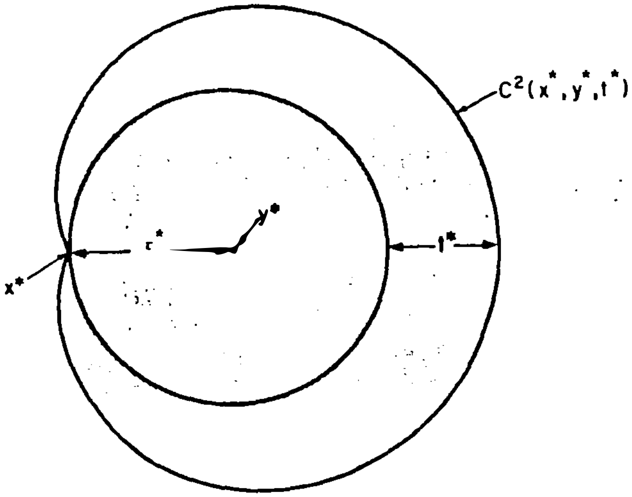


FIGURE 4 THE CARDIOD $C^2(x^*,y^*,t^*)$ (HERE $r^* = \|x^*-y^*\|$)

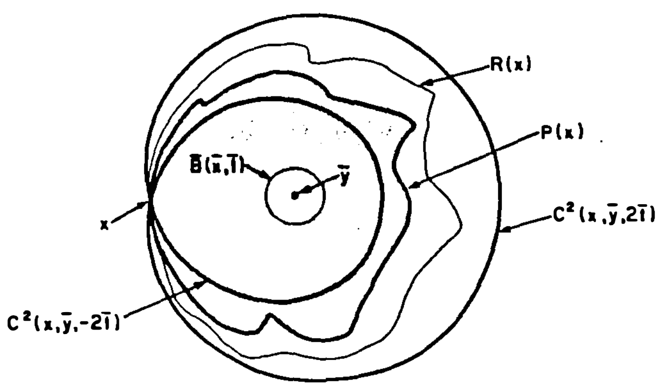


FIGURE 5 EXAMPLE OF $P(x)$ AND $R(x)$ FOR TWO DIMENSIONAL CASE

covered by a point $x \in X$. While the proposition is not our primary concern, it does have some implications worth noting. Shepsle and Weingast [1984] prove that the set of reachable points under "forward" agendas, (i.e. agendas of the form $(x_1, x_2, \ldots, x_k, x)$, where first x_k is voted against x, then the winner against x_{k-1}, etc.) is exactly $X - D^{-1}(x)$. It follows that with a sophisticated agenda one can never reach from a point x, an alternative y for which $\|y - \bar{y}\| > \|x - \bar{x}\| + 4\bar{t}$. So "forward" building sophisticated agendas are constrained in how far they can wander from the generalized median point, $\bar{y}$

Theorem 5 Under the assumptions of Proposition 8.3, $\underline{UC}(X) \subseteq \bar{B}(\bar{y}, 4\bar{t})$

Proof: For any $x \in \bar{B}(\bar{y}, 4\bar{t})$, by Prop 8.3, $\bar{y} \in D(x)$ and $\bar{y} \in P(x)$, hence $x \notin \underline{UC}(X)$.

Q.E.D.

The above proposition and theorem are illustrated in Figures 6 and 7.

The crucial parameter here is $\bar{t}$. As is described in Ferejohn, McKelvey, and Packel [1984], the size of $\bar{t}$ can be thought of as a measure of the symmetry of the distribution of ideal points. In the case when a total multidimensional median point exists, then $\bar{t}$ will be zero, since all median planes go through that point. In the more general cases, $\bar{t}$ is the radius of a smallest sphere needed to intersect all medians. So if there is just a small deviation from the

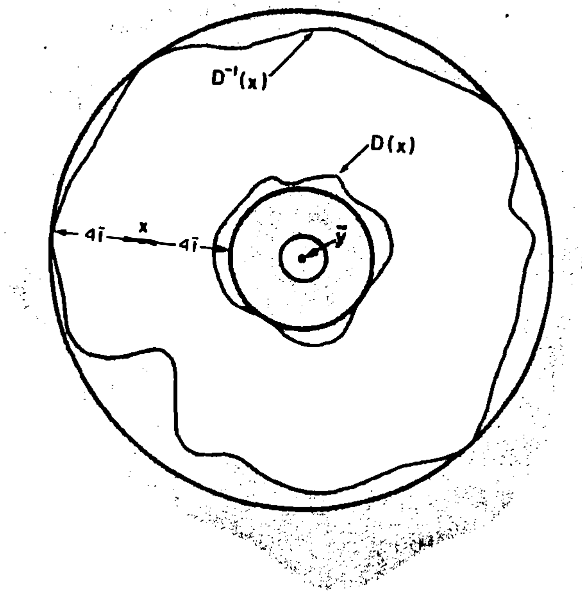


FIGURE 6 ILLUSTRATION OF Prop. 8.3 FOR TWO DIMENSIONAL EXAMPLE

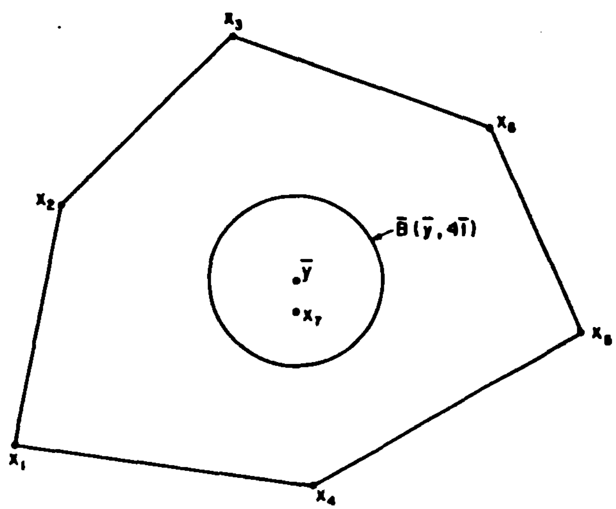


FIGURE 7 LIMITS FOR THE UNCOVERED SET FOR EXAMPLE OF FIGURE 2 $U(X) \subseteq \bar{B}(\bar{y}, 4\bar{t})$

case when a total median exists, $\bar{t}$ would be small. For larger deviations it will be larger. The smaller $\bar{t}$ is, the less latitude there is in terms of the size of the uncovered set, and the size of the set of reachable points.

As far as the location of the uncovered set, we see from the theorem, together with the observations that have been made on the size of the yolk, that at least for the "Downsian" case, the conjecture of Miller is correct. Namely the uncovered set is a centrally located set. It collapses to the core, or majority rule equilibrium when one exists, and will be small when the configuration of ideal points is perturbed slightly from a core configuration. It is centered around a generalized median set whose size is a measure of the degree of nonsymmetry of the ideal points.

9. Conclusion

We have shown that three different institutional processes all lead to points inside the uncovered set. Also, in the case of Euclidian based preferences, we have shown that the uncovered set is bounded by a sphere centered about a generalized median set--the "yolk." This set, in turn, is shown to be a centrally located set which is equal to the core, or total median when one exists, and which is small when preferences are "close" to having a core.

Other results have also identified the generalized median set, to be of particular importance. Ferejohn et al [1984] show that a Markov process, where new proposals are generated randomly to beat the previous status quo, has a limiting distribution which is centered about this set.

Although the uncovered set encompasses the equilibria for several different institutions, it should be emphasized that the uncovered set is not a catch-all. In particular, Kramer and McKelvey [1984] show that the "minmax set" is not included in the uncovered set. Further, any process which can pick Pareto dominated points—such as the "sophisticated voting equilibrium" of Kramer [1972], or the "institution induced equilibrium" of Shepsle [1979] can clearly lead to points outside the uncovered set. Even the Markov process of Ferejohn et al [1980], described above can select points outside the uncovered set. While the limiting distribution for the Markov process is centered about the generalized median set it can put some probability beyond the distance (of $4\bar{t}$) which bounds the uncovered set. A subject for future research will be to try and identify the characteristics of institutions whose equilibria lie in the uncovered set.

APPENDIX A

In this appendix, we prove the results labeled as "Propositions" in the body of the paper. We first round up a herd of lemmas, which characterize various properties of the social order, P, and which are used in the proofs of the main results. Throughout the Appendix, since assumptions A0 and B0 are always made, they will not be explicitly stated as assumptions in the results.

Lemma 1 Under Assumption A1, for any $x \in X$ and $C \subseteq N$, if $P_C(x) \neq \emptyset$, then $R_C(x) = \overline{P_C(x)}$.

Proof: We first show that

$$\bigcap_{i \in C} \overline{P_i(x)} = \overline{P_C(x)} \qquad (A.1)$$

We can use the properties of closure of intersections of sets to obtain $\overline{P_C(x)} = \overline{\bigcap_{i \in C} P_i(x)} \subseteq \bigcap_{i \in C} \overline{P_i(x)}$. To show the reverse inclusion, let $y \in \bigcap_{i \in C} \overline{P_i(x)}$. Then $y \in R_i(x)$ for all $i \in C$. But now let $L = \{z = ty + (1 - t)x \mid 0 < t < 1\}$. Then, by A1, $L \subseteq P_i(x)$ for all $i \in C$, and $N(y) \cap L \neq \emptyset$ for all neighborhoods $N(y)$ of y. Hence, $N(y) \cap P_i(x) \neq \emptyset$, so $y \in \overline{P_C(x)}$. Using (A.1), we now get

$$R_C(x) = \bigcap_{i \in C} R_i(x) = \bigcap_{i \in C} \overline{P_i(x)} = \overline{P_C(x)}.$$

Q.E.D.

Lemma 2 For all $i \in N$, P_i is an open correspondence.

<u>Proof</u> Let $u_i:X \to \mathbb{R}$ be a continuous utility representation of P_i, and let $(x_0,y_0) \in P_i$. I.e. $x_0P_iy_0$, or $u_i(x_0) > u_i(y_0)$. Set $\varepsilon = u_i(x_0) - u_i(y_0)$, and find neighborhoods $N(x_0)$ and $N(y_0)$ of x_0 and y_0 respectively such that $x \in N(x_0) \Rightarrow u_i(x) > u_i(x_0) - \frac{\varepsilon}{2}$ and $y \in N(y_0) \to u_i(y) < u_i(y_0) + \frac{\varepsilon}{2}$. Then, for $x \in N(x_0)$ and $y \in N(y_0)$,

$$u_i(x) > u_i(x_0) - \frac{\varepsilon}{2} = u_i(y_0) + \frac{\varepsilon}{2} > u_i(y)$$

So $N(x_0) \times N(y_0)$ is an open neighborhood of (x_0,y_0) contained in P_i. Hence P_i is open.

Q.E.D.

<u>Lemma 3</u> For all $k = \pm1,\pm2,\ldots$, P^k is an open correspondence.

<u>Proof</u>: For $k = 1$, we have $P^1 = P = \bigcup_{C\in \underline{W}} \bigcap_{i\in C} P_i$,

so since, by Lemma 2, each P_i is open, so is P. Now assume P^k is open, and we show that P^{k+1} is. If $(x_0,y_0) \in P^{k+1}$, then $\exists z \in X$ such that $x_0P z$ and zP^ky_0. By openness of P^k and P^1, there are neighborhoods $N(x_0)$ and $N(y_0)$ of x_0 and y_0 such that $x \in N(x_0) \Rightarrow xP z$ and $y \in N(y_0) \Rightarrow zP^ky$

But then $N(x_0) \times N(y_0) \subseteq P^{k+1}$, so P^{k+1} is open. Finally, for k negative, the result follows from the observation that $(x,y) \in P^k \Leftrightarrow (y,x) \in P^{-k}$.

Q.E.D.

<u>Lemma 4</u> Under Assumption A1, for all $k = \pm1,\pm2,\ldots$, both P^k and R^k are lower hemi continuous correspondences.

<u>Proof</u>: Lower hemi continuity of P follows directly from Lemma 3, since $P = P^1$ is an open correspondence. Then by induction lower hemi continuity of P^k follows because $P^k = P \circ P^{k-1}$ is the composition of two lower hemi continuous correspondences. Note that lower hemi continuity of P^k does not depend on assumption A1.

To show that R is lower hemi continuous, apply Lemma 1 to get, for $P(x) \neq \emptyset$, $R(x) = \bigcup_{C\in \underline{B}} R_C(x) = \bigcup_{C\in B} \overline{P_C(x)} = \overline{\bigcup_{C\in B} P_C(x)}$. For $P(x) = \emptyset$, $R(x) = \{x\}$. Since $x \in R(x)$ for $P(x) \neq \emptyset$, we can write $R(x) = \{x\} \cup \overline{\bigcup_{C\in B} P_C(x)}$. But $\bigcup_{C\in B} P_C(x)$ is open, hence l.h.c. Then, by proposition 11.19(b) of Border, $\overline{\bigcup_{C\in B} P_C(x)}$ is l.h.c. Also $\{x\}$ is l.h.c. Then since the union of two l.h.c. correspondences is l.h.c., this proves R is l.h.c. The result for R^k follows as above for P^k.

Q.E.D.

<u>Lemma 5</u> Under Assumption A1, for all $x \in X$, $P(x)$ is starlike about x. Futher, for all $y \in P(x)$, if $z = ty + (1 - t)x$, with $t < 0$, then $z \in P^{-1}(x)$.

<u>Proof</u>: To show that $P(x)$ is starlike about x, we must show that for all $y \in X$, if $y \in P(x)$, then $z = ty + (1-t)x \in P(x)$ whenever $0 < t < 1$. But by strict quasi concavity, for all $i \in N$, if $y \in P_i(x)$, then $z \in P_i(x)$. So, for any $C \subseteq N$, $yP_Cx \Rightarrow zP_Cx$. To show

the second assertion, pick $x \in X$, $y \in P(x)$ and $z = ty + (1 - t)x$ for $t < 0$. Then $x = rz + (1 - r)y$, where $r = \frac{1}{1-t}$. So $0 < r < 1$, and x is a convex combination of z and y. Now for all $i \in N$, we have $yP_ix \Rightarrow xP_iz$. To see this, suppose not. I.e. suppose yP_ix and zR_ix. By completeness of R_i, either yR_iz or zR_iy. But by strict quasi concavity, $yR_iz \Rightarrow xP_iz$, and $zR_iy \Rightarrow xP_iy$. Either of these yields a contradiction, so we must have $yP_ix \Rightarrow xP_iz$, but then for any $C \subseteq N$, $yP_Cx \Rightarrow xP_Cz$, hence $yPx \Rightarrow xPz \Rightarrow z \in P^{-1}(x)$.

Q.E.D.

Lemma 6 Under Assumption A1, for all x, if $P(x) \neq \emptyset$, then for every neighborhood, $N(x)$ of x, $N(x) \cap P(x) \neq \emptyset$ and $N(x) \cap P^{-1}(x) \neq \emptyset$.

Proof: The result follows immediately from Lemma 5.

Q.E.D.

Lemma 7 Under Assumption A1 and B1, then for all $x \in X$, if $P(x) \neq \emptyset$, $I(x) = \beta(P(x)) = \beta(P^{-1}(x))$. So $R(x) = \overline{P(x)}$.

Proof: First, it is clear that $\beta(P(x)) \subseteq I(x)$ and $\beta(P^{-1}(x)) \subseteq I(x)$. To see this, assume $y \in \beta(P(x))$. Then any neighborhood of y must intersect $P(x)$ and $X - P(x) = R^{-1}(x)$. But then $y \notin P(x)$ because $P(x)$ is open, and $y \notin P^{-1}(x)$ because $P^{-1}(x)$ is open and $P^{-1}(x) \cap P(x) = \emptyset$. But $P(x)$, $P^{-1}(x)$ and $I(x)$ partition X, so we must have $y \in I(x)$. A similar argument shows $\beta(P^{-1}(x)) \subseteq I(x)$.

Now, to show $I(x) \subseteq \beta(P(x))$ and $I(x) \subseteq \beta(P^{-1}(x))$, pick

$y \in I(x)$. If $y = x$, then the result follows from Lemma 6. If $y \neq x$, then pick $z = ty + (1 - t)x$ where $0 < t < 1$, and $w = sy + (1 - s)x$, where $s > 1$. By strict quasi concavity, we have, for all $i \in N$,

$$\begin{aligned} yR_ix &\Rightarrow zP_ix \\ xR_iy &\Rightarrow xP_iw \end{aligned} \tag{A.2}$$

But then n odd and xIy means $\{i \in N \mid xP_iy\} \notin \underline{W}$ and $\{i \in N \mid yP_ix\} \notin \underline{W}$ or, equivalently $\{i \in N \mid yR_ix\} \in \underline{B}$ and $\{i \in N \mid xR_iy\} \in \underline{B}$. By the assumption that $\underline{W}$ is strong and proper, it follows $\underline{B} = \underline{W}$, so we can replace $\underline{B}$ by $\underline{W}$ in the above expressions, but then, from equation (A.2), it follows that

$$\{i \in N \mid yR_ix\} \subseteq \{i \in N \mid zP_ix\}$$

and

$$\{i \in N \mid xR_iy\} \subseteq \{i \in N \mid xP_iw\}$$

Hence, by property (a) of $\underline{W}$, $\{i \in N \mid zP_ix\} \in \underline{W}$ and $\{i \in N \mid xP_iw\} \in \underline{W}$. So $z \in P(x)$ and $w \in P^{-1}(x)$. Since t and s are arbitrary, z and w can be chosen to be in any neighborhood $N(y)$ of y. Hence $y \in \beta(P(x))$ and $y \in \beta(P^{-1}(x))$. This completes proof of the first statement of the Lemma. The fact that $R(x) = \overline{P(x)}$ now follows immediately from the fact that $I(x) = \beta(P(x))$.

Q.E.D.

Lemma 8 Under Assumptions A1 and B1, if $x, y \in X$ with $x \neq y$, and $z = ty + (1-t)x$ with $0 < t < 1$, then $y \in R(x) \Rightarrow z \in P(x)$. Further, $R(x)$ satisfies the same properties as $P(x)$ in Lemmas 5 and 6.

Proof Let x, y, z be as described in the Lemma, and $y \in R(x)$. Then $\neg xPy$ means $\{i \in N | xP_i y\} \notin \underline{W}$ or, taking the complement and using the fact $\underline{W}$ is strong and proper, $\{i \in N | yR_i x\} \in \underline{B} = \underline{W}$

But then, by strict quasi concavity, $yR_i x \Rightarrow zP_i x$ for all i, so

$$\{i \in N | yR_i x\} \subseteq \{ i \in N | zP_i x\}$$

so, by property (a) of $\underline{W}$, $\{i \in N | zP_i x\} \in \underline{W}$. Hence zPx. The remainder of the Lemma follows by similar arguments to those of Lemmas 5 and 6.

Q.E.D.

Lemma 9 Under Assumptions A1 and B1, for all $x, y \in X$ with $x \neq y$, $P(x) \cap P(y) \neq \emptyset$ whenever $P(x)$ and $P(y)$ are non empty.

Proof: Assume, for some $x, y \in X$ that

$$P(x) \cap P(y) = \emptyset. \tag{A.3}$$

then, we must have xIy. Otherwise, if for example xPy, then by openness of $P(y)$, there is a neighborhood $N(x)$ of x such that $N(x) \subseteq P(y)$, and since $P(x) \cap P(y) = \emptyset$, we must have $N(x) \cap P(x) = \emptyset$. But this contradicts Lemma 6. A similar argument shows we cannot have yPx. So we must have xIy. But then pick $z = \frac{1}{2}x + \frac{1}{2}y$. By Lemma 8 it follows that zPx and zPy a contradiction to (A.3). So we must have $P(x) \cap P(y) \neq \emptyset$.

Q.E.D.

Proof of Proposition 3.1: We first show that D satisfies the properties stated.

If $x = y$, then $P(x) = P(y)$ and $R(x) = R(y)$, so $\neg xDy$. Hence D is irreflexive.

If $x \neq y$, and xDy, then either $P(x) \subsetneq P(y)$ or $R(x) \subsetneq R(y)$. In either case $\sim yDx$. So D is asymmetric.

If xDy and yDz then $P(x) \subseteq P(y)$ and $P(y) \subseteq P(z)$, so $P(x) \subseteq P(z)$. Also $R(x) \subseteq R(y)$ and $R(y) \subseteq R(z)$. So $R(x) \subseteq R(z)$. Also at least one of these inclusions is strict since for the inclusions $P(x) \subseteq P(y)$ and $R(x) \subseteq R(y)$, at least one is strict. So xDz.

Acyclicity follows directly from the fact that D is asymmetric and transitive.

So D satisfies the stated properties. We now show that C does.

If $x = y$, then $\sim xPy \Rightarrow \neg xCy$. So C is irreflexive.

If $x \neq y$, then $xCy \Rightarrow xPy \Rightarrow \sim yPx \Rightarrow \sim yCx$. Hence C is asymmetric.

If xCy and yCz, then $x\bar{D}y$ and $y\bar{D}z$ and xPy and yPz. By an argument similar to above, it follows $\bar{D}$ is transitive, so $x\bar{D}z$. But since $y\bar{D}z$, it follows that $P(y) \subseteq P(z)$. Thus $xPy \Rightarrow x \in P(y) \Rightarrow x \in P(z) \Rightarrow xPz$. Thus $x\bar{D}z$ and xPz. I.e., xCz. So

C is transitive.

As above, acyclicity of C follows from transitivity and asymmetry.

Finally, for any $x,y \in X$, $xCy \Rightarrow x\bar{D}y$ and $xPy \Rightarrow x \in P(y)$ and $P(x) \subseteq P(y)$. But $x \notin P(x)$, so $P(x) \subsetneq P(y)$. Hence xDy. Thus $C \subseteq D$.

Q.E.D.

<u>Proof of Proposition 3.2</u>: Let $\{y_i\}_{i=1}^{\infty}$ be a sequence with $y_i \to y^* \in X$, and with $y_i \in \bar{D}(x)$ for all i. Thus for all i, $P(y_i) \subseteq P(x)$ and $R(y_i) \subseteq R(x)$. Assume $y^* \notin \bar{D}(x)$. Then either $P(y^*) \not\subseteq P(x)$ or $R(y^*) \not\subseteq R(x)$. If $P(y^*) \not\subseteq P(x)$, then set $G = X - \overline{P(x)}$. Since $P(x)$ is open, G is open, and $G \cap P(y^*) \neq \emptyset$. Hence, by lower hemi continuity of P, Lemma 4, there is a neighborhood $N(y^*)$ of y^* such that $P(y) \cap G \neq \emptyset$ for all $y \in N(y^*)$. In particular, since $y_i \to y^*$, $P(y_i) \cap G \neq \emptyset$ for some y_i. But this contradicts $P(y_i) \subseteq P(x)$, so we must have $P(y^*) \subseteq P(x)$.

Now assume $R(y^*) \not\subseteq R(x)$. Now set $G = X - R(x)$. Again G is open and $R(y^*) \cap G \neq \emptyset$. Again, we can apply lower hemi continuity of R to get a neighborhood $N(y^*)$ of y^* for which $R(y) \cap G \neq \emptyset$ for $y \in N(y^*)$, but then $R(y_i) \cap G \neq \emptyset$ for some i, which contradicts $R(y_i) \subseteq R(x)$.

Q.E.D.

<u>Proof of Proposition 3.3</u>: $x\bar{D}y \Rightarrow P(x) \subseteq P(y)$ and $x\bar{D}y \Rightarrow R(x) \subseteq R(y)$ follows directly from the definition of $\bar{D}$, so we need only show the reverse implications. We show $P(x) \subseteq P(y) \Leftrightarrow R(x) \subseteq R(y)$. Then

clearly either one of these inclusions implies $x\bar{D}y$. By Lemma 7, $P(x) = [R(x)]^{o}$ for all x, hence $R(x) \subseteq R(y) \Rightarrow [R(x)]^{o} \subseteq [R(y)]^{o}$ $\Rightarrow P(x) \subseteq P(y)$. Now assume $P(x) \subseteq P(y)$. If $P(x) \neq \emptyset$, then Lemma 7 implies $R(x) = \overline{P(x)} \subseteq \overline{P(y)} = R(y)$. If $P(x) = \emptyset$, then Lemma 8 implies $R(x) = \{x\}$. But $P(x) = \emptyset \Rightarrow y \notin P(x) \Rightarrow x \in R(y)$ so $R(x) \subseteq R(y)$. This proves the first line of implication. The second and third lines follow directly using the definitions of D and C in terms of $\bar{D}$.

Q.E.D.

<u>Proof of Proposition 3.4</u>: For any $x,z \in X$

$$\begin{aligned} z \in P^{-2}(x) &\Leftrightarrow \exists y \in X \text{ s.t. } xPy \ \& \ yPz \\ &\Leftrightarrow \exists y \in X \text{ s.t. } y \notin R(x) \ \& \ y \in P(z) \\ &\Leftrightarrow P(z) \not\subseteq R(x) \\ &\Rightarrow P(z) \not\subseteq P(x) \\ &\Rightarrow \sim z\bar{D}x \end{aligned}$$

Now if Assumptions A1 and B1 are met, then by Lemma 7 and Prop 3.3, the last two implications become $\Leftrightarrow$.

Q.E.D.

<u>Proof of Proposition 4.1</u>: To show $\bigcap_{y \in X} P^2(y) \subseteq UD(X)$, pick $x \in \bigcap_{y \in X} P^2(y)$ (the proof is trivial if this set is empty). Then, for all $y \in X$, $y \in P^{-2}(x)$. By Proposition 3.4, $\sim y\bar{D}x$. Thus $x \in UD(X)$.

The fact that $UD(X) \subseteq UC(X)$ follows directly from $C \subseteq D$.

To show $UC(X) \subseteq \bigcap_{y \in X} R^2(y)$, let $x \in UC(X)$. Then, for all

$y \varepsilon X$, $\sim yCx$. I.e., $\sim(P(y) \subseteq P(x))$ or $\sim(R(y) \subseteq R(x))$ or $\sim yPx$. But $\sim yPx \Rightarrow xRy \Rightarrow x \varepsilon R^2(y)$. Next, $\sim(R(y) \subseteq R(x)) \Rightarrow \exists z \varepsilon X$ with zRy and $\sim zRx$. I.e., zRy and xPz, so $x \varepsilon R^2(y)$.

Finally, $\sim(P(y) \subseteq P(x)) \Rightarrow \exists z \varepsilon X$ with zPy and $\sim zPx$. I.e., zPy and xRz. So $x \varepsilon R^2(y)$.

To show the second assertion, we need only show that under A1 and B1, $\bigcap_{y \varepsilon X} R^2(y) \subseteq \overline{\bigcap_{y \varepsilon X} P^2(y)}$. First we show that $R^2(y) \subseteq \overline{P^2(y)}$, for any $y \varepsilon X$. Let $z \varepsilon R^2(y)$. Then $\exists$ $w \varepsilon X$ with wRy and zRw. But now, by Lemma 7, $R(w) = \overline{P(w)}$, so for any neighborhood $N(z)$ of z, $N(z) \cap P(w) \neq \emptyset$. By lower hemi continuity of P (Lemma 4), there is a neighborhood, $N(w)$ of w such that, for all $w' \varepsilon N(w)$, $N(z) \cap P(w') \neq \emptyset$. Now, since $R(y) = \overline{P(y)}$, we can pick $w' \varepsilon N(w)$ with $w'Py$. Then picking $z' \varepsilon P(w') \cap N(z)$, we have $z' \varepsilon P^2(y)$, so $z \varepsilon \overline{P^2(y)}$. But now, we have $\bigcap_{y \varepsilon X} R^2(y) \subseteq \bigcap_{y \varepsilon X} \overline{P^2(y)} = \overline{\bigcap_{y \varepsilon X} P^2(y)}$. To show the last equality, note that $z \varepsilon \bigcap_{y \varepsilon X} \overline{P^2(y)} \Leftrightarrow$ for all $y \varepsilon X$, and every neighborhood, $N(z)$ of z, $N(z) \cap P^2(y) = \emptyset$. $\Leftrightarrow$ for every neighborhood, $N(z)$ of z, $N(z) \cap [\bigcap_{y \varepsilon X} P^2(y)] \neq \emptyset \Leftrightarrow z \varepsilon \overline{\bigcap_{y \varepsilon X} P^2(y)}$.

Q.E.D.

<u>Proof of Proposition 4.2</u>: $x \notin \underline{M}(D,A) \Rightarrow \exists z \varepsilon A$ with zDx. But then $\{x,z\}$ is a chain, which, by the Kuratowski Lemma, is part of a maximal chain, say $E \subseteq A$. But by the proof of Theorem 1, E has an upper bound, say y. I.e., yDw or $y = w$ for all $w \varepsilon E$. It follows that

$y \varepsilon E$ but by transitivity of D, we must have $D(y) = \emptyset$. Otherwise E is not maximal.

<u>Proof of Proposition 4.3</u>: Suppose $\bigcap_j P(x^j) \cap \underline{M}(D,X) = \emptyset$, and pick $x \varepsilon \bigcap_j P(x^j)$. Then since $x \notin \underline{M}(D,X)$, by Proposition 4.2, $\exists$ $y \varepsilon \underline{M}(D,X)$ with $y\bar{D}x$. I.e. $P(y) \subseteq P(x)$ and $R(y) \subseteq R(x)$ (one strict). But since $x \varepsilon \bigcap_j P(x^j)$, we have x^j Px for all j, or $\{x^j\} \subseteq P^{-1}(x) \subseteq P^{-1}(y)$. So yPx^j for all j. But then $y \varepsilon \bigcap_j P(x^j) \cap \underline{M}(D,X)$, a contradiction. The proof of the second inequality is similar.

<u>Proof of Proposition 7.1</u>: Assume there is some $x_i \varepsilon A$ with $x_i C_A x_t^*$. I.e., $x_i Px_t^*$ and $x_i \bar{D}_A x_t^*$. By the definition of the sophisticated equivalent, it follows that $x_t^* = x_k^* = x_k$ for some $x_k \varepsilon A$. There are two cases:

<u>Case 1</u>: $i < k$ Here, from the definition of the sophisticated equivalent, since $x_k = x_k^*$, we have $x_k P x_j^*$ for all $j < k$. But since $x_i \bar{D}_A x_k$, it follows that for $j < i$, $x_i P x_j^*$. Otherwise $x_j^* \varepsilon R(x_i) \Rightarrow x_j^* \varepsilon R(x_k) \Rightarrow \sim x_k P x_j^*$, a contradiction. But then, by the definition of the sophisticated equivalent, $x_i^* = x_i$. But $x_k^* = x_k$ and $i < k \Rightarrow x_k P x_i^* \Rightarrow x_k P x_i$, a contradiction.

<u>Case 2</u>: $i > k$. Now, since $x_k = x_k^*$, $x_k P x_j^*$ for all $j < k$. By $x_i \bar{D}_A x_k$, it follows that $x_i P x_j^*$ for all $j < k$. But for $k \leq j < i$, $x_j^* = x_k$, since $x_n^* = x_k$. But then $x_i P x_k \Rightarrow x_i P x_j^*$ for $k \leq j < i$. Hence $x_i P x_j^*$ for all $j < i$, which means, by the definition of the sophisticated

equivalent, that $x_i^* = x_i$. But this is a contradiction to $x_n^* = x_k$, since this would imply $x_i^* = x_k$ also.

So, in both cases, we get a contradiction, implying that for any $x_k \in A$, we cannot have $x_k P x_t^*$ and $x_k \bar{D}_A x_t^*$. It follows that $\sim x_k C_A x_t^*$, so $x_t^* \in \underline{M}(C_A, A)$. This proves the first assertion; the second follows immediately from the observation that if P is anti symmetric, then $xD_A x_t \Rightarrow xPx_t$, hence by the first part of the theorem, $xPx_t \Rightarrow \sim x\bar{D}x_t \Rightarrow \sim xD_A x_t$, a contradiction. So for all $x \in A, \sim xD_A x_t$. I.e., $x_t \in \underline{M}(D_A, A)$.

Q.E.D.

APPENDIX B

This appendix provides formal definitions of the sets $C^m(x,y,t)$ as well as a proof of the propositions of section 8. Further details can be found in Ferejohn, McKelvey, and Packel [1981].

Let $\theta(x) = (\theta_1(x), \theta_2(x), \ldots, \theta_{m-1}(x), \rho(x))$ denote the m dimensional, spherical coordinates of the vector $x \in R^m$. Thus

$$\rho(x) = ||x|| \tag{B1}$$

and, for $1 \leq i \leq m-1$,

$$\theta_i(x) = \sin^{-1}\left[\frac{x_i}{\rho(x) \prod_{j<i} \cos\theta_j(x)}\right] \tag{B2}$$

Here, the θ_j range between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, except for θ_{n-1}, which ranges between $-\frac{\pi}{2}$ and $3\frac{\pi}{2}$. Now for any $x^*, y^* \in R^m$, and $t^* \in R$, we set $r^* = ||x^* - y^*||$, and let Q be an m x m orthonormal rotation matrix such that

$$Q(y^* - x^*) = (t, 0, \ldots, 0). \tag{B3}$$

Write $Q(x-x^*) = (z_1, \ldots, z_m) = z$. Then define $\zeta_{x^*,y^*}(x) = \theta(Q(y^* - x^*)) = \theta(z)$. So $\zeta_{x^*,y^*}(x)$ are m dimensional spherical coordinates of x which are centered at x^* and have one axis coincident with the vector $y^* - x^*$. Now set

$$\alpha = \sin^{-1} \frac{-t^*}{r^*} \text{ where } -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}. \tag{B4}$$

So that the above is also well defined for the case when $|t^*| \geq r^*$, we use the convention that, for $r \in \mathbb{R}$, $|r| \geq 1$, $\sin^{-1} r = \frac{\pi}{2} \operatorname{sgn} r$. We then set

$$\beta = \begin{cases} \frac{\pi}{2} & \text{if } m > 2 \\ \pi - \alpha & \text{if } m = 2 \end{cases} \tag{B5}$$

Then define, for $t^* \in \mathbb{R}$,

$$C^m(x^*, y^*, t^*) = \{x \in X \mid 0 \leq \rho(z) \leq 2r^* \sin \theta_1(z) + t^* \text{ and } \alpha \leq \theta_1(z) \leq \beta \text{ (where } z = Q(x - x^*))\} \tag{B6}$$

Thus, $C^m(x^*, y^*, t^*)$ is the m dimensional cardioid with cusp at x^*, center at y^*, eccentricity of t^*, and radius of r^*. Note that if $t^* = 0$, then $C^m(x^*, y^*, t^*)$ becomes a sphere, with center at y^* and radius r^*. If $t^* < 0$, then the resulting cardioid is contained in this sphere, otherwise it contains the sphere. Also note that if $t^* < -r^*$, then $C^m(x^*, y^*, t^*) = \emptyset$.

Proposition 8.1 Let Assumptions A3 and B2 be met. Let $\bar{y} \in X$ and $\bar{r} \in \mathbb{R}^+$ be chosen so that for every median hyperplane $H(a,c)$, with $a \in \mathbb{R}^m$, $c \in \mathbb{R}$, that $H(a,c) \cap \bar{B}(\bar{y}, \bar{r}) \neq \emptyset$. Then

$$C^m(x, \bar{y}, -2\bar{r}) \subseteq P(x) \subseteq R(x) \subseteq C^m(x, \bar{y}, 2\bar{r})$$

Proof We write B_0 for $\bar{B}(\bar{y}, \bar{r})$, and $\underline{M}$ for the set of all median hyperplanes. Pick $x \in \mathbb{R}^m$, and set $r = \|x - \bar{y}\|$. We choose coordinates so that B_0 is centered at $(r, 0, \dots, 0)$, (which translates to $(\frac{\pi}{2}, 0, \dots, 0, r)$ in spherical coordinates), and so that x is at the origin. Now θ_1 (actually $\frac{\pi}{2} - \theta_1$) measures the angle an arbitrary point y, with spherical coordinates $(\theta_1, \dots, \theta_{n-1}, \rho)$ makes with the axis between the origin and the center of B_0. We consider the points on the ray from the origin thru y, and characterize those points on the ray which are in $P(x)$ and $R(x)$. We assume y is of unit length, so $\rho(y) = \|y\| = 1$. A point on the ray is of the form λy, with $\lambda > 0$, and has spherical coordinates which are the same as those of y, except $\rho(\lambda y) = \lambda$.

First note that the set of median hyperplanes in the direction y is a closed set. I.e., $\{c \in \mathbb{R} \mid H(y, c) \in \underline{M}\}$ is closed. So we let c_L and c_H be the inf and sup of $\{c \in \mathbb{R} \mid H(y, c) \in \underline{M}\}$, and set $H_L = H(y, c_L)$ and $H_H = H(y, c_H)$. Of course if n is odd, $H_L = H_H$. By virtue of Assumption A3, we get, for $\lambda > 0$,

$$\begin{aligned} \lambda y \in P(x) &\iff \lambda < 2c_L \\ \lambda y \in R(x) &\iff \lambda \leq 2c_H \end{aligned} \tag{B7}$$

But, by assumption of the Lemma $\{c_L, c_H\} \subseteq \{c \in \mathbb{R} \mid H(y,c) \cap B_0 \neq \emptyset\}$. Letting b_L and b_H be the inf and sup of this latter set, we get

$$b_L \leq c_L \leq c_H \leq b_H \tag{B8}$$

Consequently

$$\lambda < 2b_L \Rightarrow \lambda < 2c_L \Rightarrow \lambda y \in P(x)$$
$$\text{and} \tag{B9}$$
$$\lambda y \in R(x) \Rightarrow \lambda \leq 2c_H \Rightarrow \lambda \leq 2b_H$$

But, by construction, b_L and b_H are obtained simply by projecting the center, $\bar{y}$, of $\bar{B}$ on y, and then adding or subtracting the radius, $\bar{r}$ of $\bar{B}$. I.e.,

$$b_L = r \sin \theta_1 - \bar{r}$$
$$\tag{B10}$$
$$b_H = r \sin \theta_1 + \bar{r}.$$

So, since $\lambda = \rho(\lambda y) = \lambda\rho(y)$, we have for any $w = \lambda y$, $\lambda > 0$.

$$\rho(w) < 2r \sin \theta_1(w) - 2\bar{r} \Rightarrow w \in P(x)$$
$$\tag{B11}$$
$$w \in R(x) \Rightarrow \rho(w) \leq 2r \sin \theta_1(w) + 2\bar{r}.$$

Now, applying definition (B6), we get for any $w = \lambda y$, $\lambda > 0$.

$$w \in C^m(x,\bar{y},-2\bar{r}) \Rightarrow w \in P(x)$$
$$\tag{B12}$$
$$w \in R(x) \Rightarrow w \in C^m(x,\bar{y},2\bar{r})$$

Since y is an arbitrary unit length vector, the result of the Lemma follows directly.

Q.E.D.

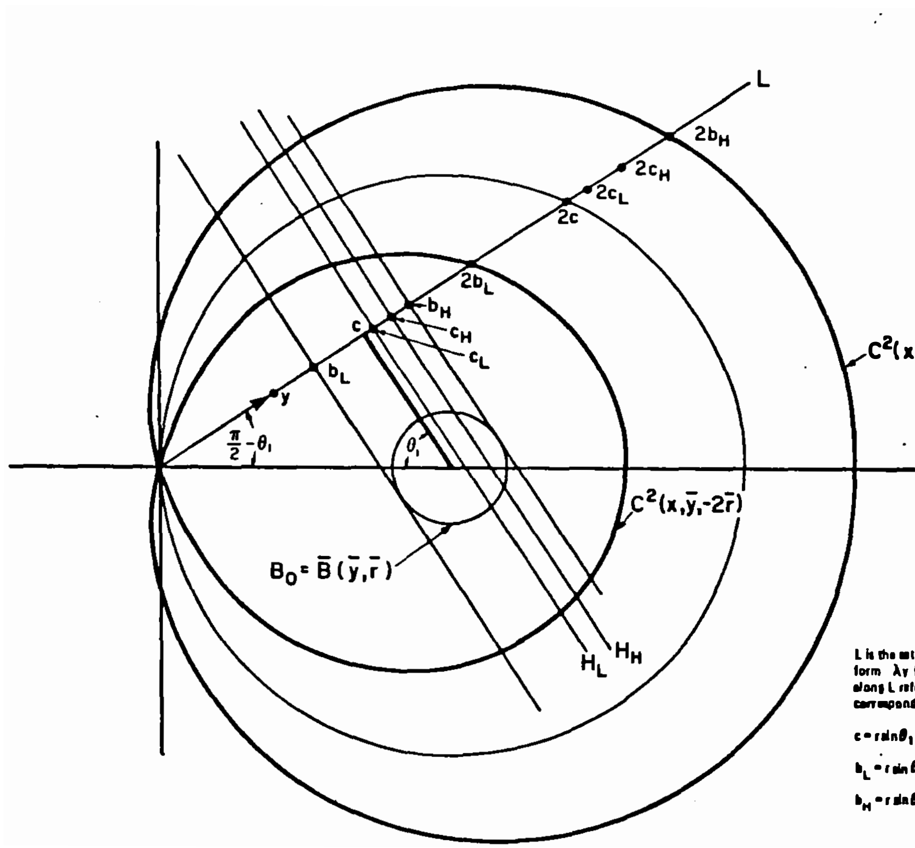
Figure A1 illustrates the construction of Lemma 8.1 for the two dimensional case.

<u>Proof of Proposition 8.3</u>: Let $y \in \bar{B}(\bar{y}, t - 4\bar{t})$. So $||y - \bar{y}|| \leq ||x - \bar{y}|| - 4\bar{t}$. Then from Lemmas 8.1 and 8.2,

$$P(y) \subseteq C^m(y,\bar{y},2\bar{t}) \subseteq \bar{B}(\bar{y},||y - \bar{y}|| + 2\bar{t})$$
$$\subseteq B(\bar{y},||x - \bar{y}|| - 2\bar{t}) \subseteq C^m(x,\bar{y}, - 2\bar{t}) \subseteq P(x).$$

So $P(y) \subseteq P(x)$, and a similar chain shows $R(y) \subseteq R(x)$. Also $y \in P(x)$ and $y \notin P(y)$, so the inclusion $P(y) \subseteq P(x)$ is strict, hence $y \in D(x)$, as we wished to show. The second assertion follows in exactly analogous argument.

Q.E.D.



L is the set of all vectors of the form λy for $\lambda > 0$. Markings along L refer to the length of the corresponding vector.

$c = r\sin\theta_1$

$b_L = r\sin\theta_1 - \bar{r}$

$b_H = r\sin\theta_1 + \bar{r}$

FIGURE A1 ILLUSTRATION OF CONSTRUCTION FOR LEMMA 1

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